Metric Spaces and Topology
Lecture 19

Metric spaces as topological spaces. For a metric space (X, d), its collection $\mathcal{T}_d$ of open sets form a topology as we have shown.
We have also shown:

(i) open balls form a basis, in fact, open balls of radii $\frac{1}{n}$, $n \in \mathbb{N}^+$, also form a basis.

(ii) for each $x \in X$, the balls $B_{\frac{1}{n}}(x)$, $n \in \mathbb{N}^+$, form a neighbourhood basis at x (indeed, if $U \ni x$ is open then $B_{\frac{1}{n}}(x) \subseteq U$ for some $n \in \mathbb{N}^+$).

A topology $\mathcal{T}$ on a set X is called metrizable if it is induced by some metric, i.e. $\mathcal{T} = \mathcal{T}_d$ for some metric d.

Def. A top. space is called:

(a) 1^{st} ctbl if every point admits a ctbl neighbourhood basis.

(b) 2^{nd} ctbl if it has a ctbl basis,

(c) separable if it has a ctbl dense subset, where a subset D of a top. space X is called dense if it intersects every nonempty open set.

Point (ii) above shows:

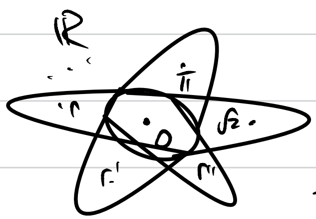
Obs. Metric spaces are 1^{st} ctbl.

Recall that:

Obs. 2^{nd} ctbl spaces are separable.

The converse is not true in general:

Counterexample. $X := \mathbb{R}$ but with the following topology:



The basis of this top is the set $\{0\}$ and sets of the form $\{0, r\}$ for each $r \in \mathbb{R} \setminus \{0\}$.

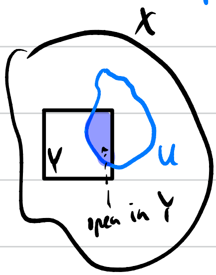
Then $\{0\}$ is dense, so X is separable, but it's not 2^{nd} ctbl because each set $\{0, r\}$ would need to be in every basis. Note that this space is actually 1^{st} ctbl.

However, recall:

Prop. In metric spaces, 2^{nd} ctbl $\iff$ separable.

Proof. $\Leftarrow$. If $D \subseteq X$ is ctbl dense, then $\{B_{\frac{1}{n}}(x) : x \in D, n \in \mathbb{N}^+\}$ is a ctbl basis. $\square$

Relative topology. Let $(X, \mathcal{T})$ be a top space and $Y \subseteq X$. The relative top $\mathcal{T}_Y$ on Y is



$$\mathcal{T}_Y := \{ U \cap Y : U \in \mathcal{T} \}.$$

Note that a relatively open set in Y may not be open in X. Some concepts descend from X to Y, for example: 1^{st} and 2^{nd} countability, but not separability: indeed, in the flower example above, the relative top on $\mathbb{R} \setminus \{0\}$ is the discrete topology, so $\mathbb{R} \setminus \{0\}$ is the only dense set and it's unctbl. However, in metric spaces, separability is hereditary because it's the same as 2^{nd} countability. Also metrizability itself is hereditary.

Criterion for basis. For a set X, a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for the topology it generates $\iff$ $\mathcal{B}$ covers X (i.e. $X = \bigcup \mathcal{B}$) and $\forall\, U, V \in \mathcal{B}$ and $x \in U \cap V$, $\exists\, W \in \mathcal{B}$ s.t. $x \in W \subseteq U \cap V$.

Proof. $\boxed{\text{HW}}$

Examples of (pre) bases. (a) For $\mathbb{R}$ with the usual topology, the following form a basis:

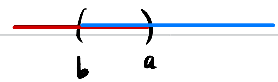
(i) open intervals

(ii) open intervals with rational endpts

(iii) for any dense $D \subseteq \mathbb{R}$, open intervals with endpts in D.

In particular, $\mathbb{R}$ is 2^{nd} cttd.

The open half lines $(-\infty, a), (b, \infty)$ form a prebasis.



(b) In $\mathbb{R}^2$ with the usual (Euclidean topology), the following form a basis:

(i) open balls

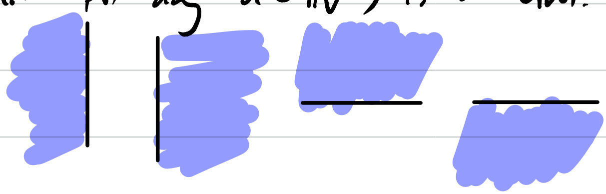
(ii) open balls with centers in $\mathbb{Q}^2$ and radii $\frac{1}{n}$, $n \in \mathbb{N}^+$.

(iii) open rectangles: $I \times J$, for open intervals $I, J \subseteq \mathbb{R}$.

(iv) open squares

In particular $\mathbb{R}^2$ (also $\mathbb{R}^d$ for any $d \in \mathbb{N}^+$) is 2^{nd} cttbl.

The orthogonal open half planes form a prebasis.



(c) In $\mathbb{R}^d$, balls, also open rectangles $I_1 \times I_2 \times \dots \times I_d$ form a basis.

(d) In $A^{\mathbb{N}}$, for any set A, the cylinders $[w]$, where $w \in A^{<\mathbb{N}}$, form a basis of clopen sets. Top. spaces that admit a basis of clopen sets are called zero-dimensional. Also, for $i \in \mathbb{N}$, $a \in A$, the sets

$$[i \mapsto a] := \{x \in A^{\mathbb{N}} : x(i) = a\}$$

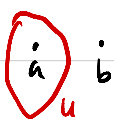
form a prebasis. (Indeed, the cylinders $[w]$ are finite intersections $[0 \mapsto w_0] \cap [1 \mapsto w_1] \cap \dots \cap [n-1 \mapsto w_{n-1}]$, for any $w \in A^n$.)

(e) In $\mathbb{Q}$ with the relative top of $\mathbb{R}$, of course the open intervals with rational endpoints form a basis, but also, the sets $[a,b] \cap \mathbb{Q}$, $a,b \in \mathbb{R} \setminus \mathbb{Q}$, are clopen (because $[a,b] \cap \mathbb{Q} = (a,b) \cap \mathbb{Q}$) and form a basis. Thus, $\mathbb{Q}$ is 0-dimensional.

Separation axioms. A top. space X is called:

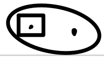
T_0: 

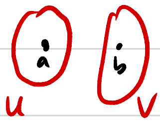
$\forall$ distinct $a,b \in X$, $\exists$ open neighbourhood U of one that doesn't contain the other.

T_1 (points are closed): 

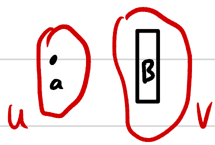
$\forall$ distinct $a,b \in X$ $\exists$ open $U \ni a$ but not b.

Obs. $T_1 \iff$ every singleton $\{a\}$ is closed. $\boxed{\text{HW}}$

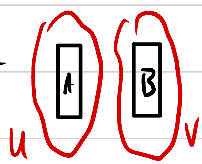
Counterexample. This space  is T_0 but not T_1.
Also, the flower example above is T_0 but not T_1.

T_2 (Hausdorff): 

for any distinct $a, b \in X$, $\exists$ disjoint open $U \ni a$ and $V \ni b$.

T_3 (regular): T_1 + 

it's T_1 and for any pt $a \in X$ and closed $B \subseteq X$ not containing a, $\exists$ disjoint open $U \ni a$ and $V \supseteq B$.

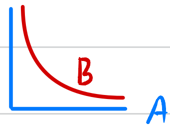
T_4 (normal): T_1 + 

it's T_1 and for any disjoint closed sets $A, B \subseteq X$, there are disjoint open sets $U \supseteq A$ and $V \supseteq B$.

It's clear that metric spaces are Hausdorff, but in fact:

Prop. Metric spaces are normal.
Proof. Let (X, d) be a metric space and $A, B \subseteq X$ be disjoint

closed sets. We don't want to look at $d(A,B) := \inf_{a\in A,\, b\in B} d(a,b)$ because that might be 0:



Take $U := \bigcup_{a\in A} B_{r_a}(a)$, where

$$r_a := \frac{1}{2} d(a,B) \quad \text{and} \quad V := \bigcup_{b\in B} B_{r'_b}(b), \; r'_b := \frac{1}{2} d(b,A). \quad \square$$

Lemma. In a normal top space X, for open sets U, V, if $U \subseteq_c V$ (i.e. $\overline{U} \subseteq V$) then $\exists$ open W s.t. $U \subseteq_c W \subseteq_c V$ (i.e. $\overline{U} \subseteq W$ and $\overline{W} \subseteq V$).

Proof. $\boxed{\text{HW}}$

Example of T_1 but not T_2. The cofinite top on an infinite set X is T_1 (each point is closed) but not T_2: indeed, any two nonempty open sets intersect. Same is true for Zariski top. on $\mathbb{F}^n$, for any infinite field $\mathbb{F}$.

Furstenberg's topology on $\mathbb{Z}$ (profinite topology). We let $\mathcal{T}_F$ be the top on $\mathbb{Z}$ generated by the arithmetic progressions, more precisely sets of the form $a + b\cdot\mathbb{Z}$, $a, b \in \mathbb{Z}, b \neq 0$,

$$a + b\cdot\mathbb{Z} := \{a + b\cdot z : z \in \mathbb{Z}\}.$$

These are exactly the subgroups of $(\mathbb{Z},+)$ and their cosets.

Claim 1. These sets form a basis.

Proof. optional $\boxed{\text{HW}}$

Claim 2. This topology is invariant under translation and $-$,
i.e. if U is open, then $t+U$ and $-U$ are open $\forall t \in \mathbb{Z}$.

Proof. This holds for the sets $a+b\mathbb{Z}$, which generate the topology. $\square$

Claim 3. This space is Hausdorff.

Proof. Let $a, b \in \mathbb{Z}$ be distinct. The top. is invariant under translation, we may assume WLOG that $a=0$.
Let $d := b+1$, then $0 \in d\mathbb{Z}$ disjoint from $b+d\mathbb{Z} \ni b$. $\square$

Claim 4. Furstenberg topology is metrizable.

Proof. Take for example $d(a,b) := \|a-b\|_2$, where
for $c \in \mathbb{Z}$, $\|c\|_2 := \sum_{\substack{n \nmid c \\ n \geq 1}} \frac{1}{2^n}$. optional $\boxed{\text{HW}}$

Claim 5. This is a 0-dim space since the sets $a+b\mathbb{Z}$ are clopen.

Proof. Indeed, the complement of $a+b\mathbb{Z}$ is a union of sets $a'+b\mathbb{Z}$ where $a' \neq a \mod |b|$, i.e. the remainders of a' and are different when divided by $|b|$. Thus, the sets $a+b\mathbb{Z}$ are clopen and they form a basis by Claim 1. $\square$

Note that $\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p \text{ prime}} p \cdot \mathbb{Z}$ because every $a \in \mathbb{Z} \setminus \{\pm 1\}$ is divisible by a prime.

Cor. $\exists$ infinitely many primes.

Proof (Furstenberg). Suppose otherwise that $\exists$ only finitely many primes $p_1, p_2, \dots, p_n$. Then $\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{i=1}^{n} p_i \mathbb{Z}$ is clopen, so the set $\{\pm 1\}$ is open, but each nonempty open set in this topology is infinite (contains a set of the form $a+b\mathbb{Z}$), by Claim 1. ☺